

MICZ-KEPLER = DYNAMICS ON THE CONE OVER THE ROTATION GROUP

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ABSTRACT. We show that the n -dimensional MICZ-Kepler system arises from symplectic reduction of a simple mechanical system on the cone over the rotation group $SO(n)$. As a corollary we derive an elementary formula for its general solution. The punch-line of our computation is that the additional MICZ-Kepler $|\phi|^2/r^2$ type potential term is the rotational part of the cone's kinetic energy.

1. INTRODUCTION

The classical mechanical formulation of a Hydrogen atom is identical to the Kepler problem for a planet moving in the gravitational field of a massive Sun. The MICZ [McIntosh-Cisneros-Zwanziger] -Kepler system is an integrable extension of the Kepler problem in which the charged proton at the origin of the Hydrogen atom is simultaneously a Dirac monopole. The original references are [9, 4]. See [2, 3] for history, references, and the n -dimensional generalization. Our purpose is to show that the MICZ-Kepler system is the reduction of a natural mechanical system (no magnetic fields) on the cone over the rotation group. See theorem 1 below and eq (8). We then use this realization to write down an explicit Lie-theoretic formula (eq 9) for the system's general solution.

Meng [3] formulated the MICZ-Kepler system as a system on what he called a 'Sternberg phase space' -and what we will call the 'adjoint bundle phase space' - associated to a principal $SO(n-1)$ bundle over $\mathbb{R}^n \setminus \{0\}$. This phase space arises as the symplectic reduction of the cotangent bundles of the same principal bundle. See [5, 6, 8]. It follows that there is a Hamiltonian system whose configuration space is Meng's principal bundle and whose reduction yields the MICZ-Kepler systems. Our object is to find this system, and then use it. The heart of the computation (noticing that the Hamiltonian eq(8) becomes (14) in adjoint bundle variables) is the observation that the angular part of the Kepler kinetic energy plus the quadratic color charge term of the MICZ-Keple Hamiltonian equal the kinetic energy for the bi-invariant metric on the full rotation group. (For the quantum-mechanical analogue of this computation compare the algebra around eq (14) to the algebra at the end of section 2 of [9].)

2. MICZ-KEPLER

The MICZ -Kepler system of equations can be written as a mixed 1st-2nd order system for a curve $(q(t), \phi(t))$ in a vector bundle over $\mathbb{R}^n \setminus \{0\}$. Here $q(t)$ denotes

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the curve in \mathbb{R}^n and $\phi(t)$ lies in the moving fiber over this curve. This fiber is the Lie algebra $\mathfrak{so}(n-1)$ of $SO(n-1)$. The bundle is called the adjoint bundle, and is an associated vector bundle to a certain principal bundle $SO(n-1) \rightarrow Q \rightarrow \mathbb{R}^n \setminus \{0\}$, called the Dirac monopole and described in the next section. So, relative to a local trivialization of the bundle, $\phi(t)$ takes values in the Lie algebra $\mathfrak{so}(n-1)$. The principal bundle is endowed with a canonical connection whose curvature is F , and which induces a covariant derivative D on the adjoint bundle. Then the MICZ-Kepler system is

$$\ddot{q} = -\frac{q}{r^3} + \frac{|\phi|^2 q}{r^4} + \phi \cdot F(\dot{q}, \cdot)$$

$$D\phi/dt = 0.$$

Some clarifications are in order. The curvature is a two-form with values in the adjoint bundle. Consequently $F(\dot{q}, \cdot)$ is a one-form with values in the adjoint bundle. The Killing form endows the adjoint bundle with a natural fiber inner product, so that $\phi \cdot F(\dot{q}, \cdot)$ is a one-form on \mathbb{R}^n . We turn this one-form into a vector using the standard flat metric. In coordinates then: $\phi \cdot F(\dot{q}, \cdot)^j = \phi_a F_{ij}^a(q) \dot{q}^i$. In a local trivialization the second equation reads $D\phi/dt = d\phi/dt + [A_i \dot{q}^i, \phi]$ where A is the connection one-form, an $\mathfrak{so}(n-1)$ -valued one-form, in this trivialization.

The system has two basic conserved quantities: its energy

$$(1) \quad H = \frac{1}{2}(|\dot{q}|^2 + \frac{|\phi|^2}{r^2}) - \frac{1}{r},$$

and the square norm of the adjoint variable, or $\mathfrak{so}(n-1)$ -

$$\text{Casimir} = \|\phi(t)\|^2.$$

There are other conserved quantities, an angular momentum and a Runge-Lenz (or Laplace) vector, but we will not need them here.

The adjoint bundle canonically fibers into adjoint-orbit fibers which are preserved by parallel transport. Consequently the fiber variable $\phi(t)$ stays on whichever adjoint orbit fiber it begins on at time 0. As a particular case, the zero-orbit is preserved and corresponds to setting $\phi = 0$ in the equations, which reduces them to the standard equations of the Kepler problem. Meng [3] takes ϕ to lie in an adjoint orbit of a particular type which he calls “magnetic”. We allow any ϕ , hence any adjoint orbit.

3. THE N-DIMENSIONAL DIRAC MONOPOLE

We describe the bundle-with-connection needed to define the MICZ-Kepler equations of the previous section. We take our description from Meng, who called it the Dirac monopole, since that is what it is when $n = 3$.

Consider the space C_0 of all orthogonal frames $f = (f_1, f_2, \dots, f_n)$ on \mathbb{R}^n normalized so that their lengths are all equal: $|f_i| = |f_j|$ all i, j . The map

$$f \mapsto f_n, \quad C_0 \rightarrow \mathbb{R}^n \setminus \{0\},$$

gives C_0 the structure of a principal $SO(n-1)$ bundle over $\mathbb{R}^n \setminus \{0\}$. Write $r = |f_n|$. Then, we have an $SO(n)$ -equivariant diffeomorphism

$$\mathbb{R}^+ \times SO(n) \rightarrow C_0; (r, g) \mapsto (rge_1, rge_2, \dots, rge_n) = (f_1, \dots, f_n),$$

where e_1, \dots, e_n is the standard basis of \mathbb{R}^n . From this perspective, the bundle projection becomes $(r, g) \mapsto rge_n$.

Put spherical coordinates on \mathbb{R}^n so that $\mathbb{R}^n \setminus \{0\} \cong \mathbb{R}^+ \times S^{n-1}$. Recall that $S^{n-1} = SO(n)/SO(n-1)$ where the $SO(n-1) \subset SO(n)$ is the stabilizer of e_n . The restriction of the principal bundle $C_0 \rightarrow \mathbb{R}^n \setminus \{0\}$ to $S^{n-1} \subset \mathbb{R}^n \setminus \{0\}$ defines the homogeneous principal bundle $\pi_{S^{n-1}} : SO(n) \rightarrow S^{n-1}$. The Killing form on $so(n)$ endows $SO(n)$ with a bi-invariant metric $d^2 s_{SO(n)}$, and relative to this metric, the orthogonal complement to the fibers of the projection $\pi_{S^{n-1}}$ define an $SO(n)$ -equivariant connection for the homogeneous principal bundle. Extend this connection trivially in the radial (r) direction to arrive at the “Dirac connection” of $C_0 \rightarrow \mathbb{R}^n \setminus \{0\}$. We refer to the bundle C_0 endowed with this connection as ‘the monopole’. For coordinate expressions for the connection see [9, 4, 2, 3].

4. ADJOINT AND CO-ADJOINT BUNDLES.

Suppose that $G \rightarrow Q \rightarrow S$ is a principal G bundle. Then the adjoint bundle $Ad(Q) := Q \times_G \mathfrak{g} \rightarrow S$ is the vector bundle associated to Q via the adjoint action of G on \mathfrak{g} . This means we divide the product $Q \times \mathfrak{g}$ by the G -induced equivalence relation $(q, \xi) \sim (qg, Ad_{g^{-1}}\xi)$. Write equivalence classes $[q, \xi]$. The fiber of the adjoint bundle is \mathfrak{g} .

The co-adjoint bundle is defined similarly, using the co-adjoint action. If \mathfrak{g} is endowed with a bi-invariant inner product such as the Killing form on $\mathfrak{g} = so(n-1)$, then we get G -equivariant isomorphisms $\mathfrak{g} \cong \mathfrak{g}^*$ and hence we can identify the adjoint bundle with the co-adjoint bundle. This identification sends adjoint orbit fiber to the corresponding co-adjoint orbit fiber. We make this identification throughout the paper. Thus the variable $\phi(t)$ in the MICZ-Kepler system is a section of the adjoint bundle $Ad(C_0)$ along the curve $q(t) \in \mathbb{R}^n \setminus \{0\}$.

A connection on Q induces on any associated vector bundle, and so on the adjoint bundle. We can describe parallel transport in $\mathfrak{g}(Q)$ along a curve $c(t) \in S$ as follows. Pick a point $\phi(0) = [q(0), \xi] \in \mathfrak{g}(Q)$. Consider the horizontal lift $q(t) \in Q$ of $c(t)$. Then $\phi(t) = [q(t), \xi] \in \mathfrak{g}(Q)$ is the parallel transport of $\phi(0)$ along $c(t)$. The 2nd MICZ-Kepler equation states that $\phi(t)$ is covariantly constant, and hence of the form just described.

5. METRIC CONES AND RIEMANNIAN SUBMERSIONS

Let $r = |f_n| \rightarrow 0$ in the construction of $C_0 \cong \mathbb{R}^+ \times SO(n)$ so that the whole of $SO(n)$ is crunched to a point defined by $r = 0$. We have formed the cone $C = Cone(SO(n)) \supset C_0$ over the full rotation group $SO(n)$. The bundle projection $C_0 \rightarrow \mathbb{R}^n \setminus \{0\}$ extends to the cone point $r = 0$, sending it to the origin.

We now put a canonical metric on the cone C .

Recall that the cone over a Riemannian manifold $(X, d^2 s_X)$ is given by the metric $dr^2 + r^2 d^2 s_X$ on $\mathbb{R}^+ \times X$ where $r \in \mathbb{R}^+$. As $r \rightarrow 0$ the metric factor involving X shrinks to 0 so it makes sense to identify $\{0\} \times X$ to a single point, which is the cone point. In this we get a metric on $Cone(X) = [0, \infty) \times X/\{0\} \times X$ which is Riemannian away from the cone point.

We form the metric cone $C = Cone(SO(n))$ by applying this cone construction to $X = SO(n)$ endowed with a Killing induced bi-invariant metric $d^2 s_{SO(n)}$. Such a metric is well-defined up to scale. That scale will be fixed by insisting that the bundle projection $C_0 \rightarrow \mathbb{R}^n \setminus \{0\}$ is a Riemannian submersion.

Recall the notion of a Riemannian submersion $\pi : Y \rightarrow S$ between Riemannian manifolds Y, S . Suppose that π is a submersion. Take any point $y \in Y$ and consider

the orthogonal complement at y to the fiber $\pi^{-1}(s)$ through y . Here $s = \pi(y)$. Following the principal bundle language as above, call this orthogonal complement $\mathcal{H}_y \subset T_y Y$ the ‘horizontal space’ at y . The differential $d\pi_y$ of π , restricted to the horizontal space, is necessarily a linear bijection onto the tangent space to s . Now the horizontal space inherits an inner product from Y . If this restricted differential is an isometry between inner product spaces for all y , then π is said to be a Riemannian submersion.

Consider the standard metric $d^2 s_{S^{n-1}}$ on the unit sphere S^{n-1} . The Killing scale for $SO(n)$ is now fixed by insisting that the bundle projection $SO(n) \rightarrow S^{n-1}$ be a Riemannian submersion. This scaling is the one for which the standard basis elements $e_i \wedge e_j$ of $so(n)$ have length 1. Here $e_i \wedge e_j$ is the skew-symmetric operator sending e_i to e_j and e_j to $-e_i$, $i \neq j$. Let θ_{ij} be the dual basis, viewed as left-invariant one-forms. Then $d^2 s_{SO(n)} = \Sigma(\theta_{ij})^2$ so that

$$(2) \quad d^2 s_C = dr^2 + r^2 \Sigma(\theta_{ij})^2$$

We verify that $SO(n) \rightarrow S^{n-1}$ and $C_0 \rightarrow \mathbb{R}^n \setminus \{0\}$ are Riemannian submersions. The connection form for the Dirac monopole is the $so(n-1)$ -valued one-form

$$(3) \quad A = \Sigma_{i < j < n} \theta_{ij} e_i \wedge e_j, \quad (\text{on } SO(n) \text{ or } C_0).$$

(Note we must use *left* invariant one forms, rather than right-invariant forms as can be seen by the fact that the connection is not G -invariant, but rather G -equivariant with G acting on the Lie algebra by the adjoint action.) The horizontal distribution is defined by $\theta_{ij} = 0, i, j \neq n$. The vertical distribution is defined by $\theta_{in} = 0, i = 1, 2, \dots, n-1$, together with $dr = 0$ in the C_0 case. Thus the Killing metric on $SO(n)$ splits orthogonally relative to the horizontal-vertical splitting

$$(4) \quad d^2 s_{SO(n)} = \Sigma(\theta_{in})^2 + \Sigma_{i < j < n} (\theta_{ij})^2$$

$$(5) \quad = \pi^* d^2 s_{S^{n-1}} + d^2 s_{fiber}$$

The fact that $\exp(te_i \wedge e_n)$ has period 2π shows that the metric scalings are correct for the Riemannian submersion: a 2π periodic horizontal geodesic in $SO(n)$ maps to a great circle of circumference 2π in S^{n-1} .

Now, written out in spherical coordinates the metric on \mathbb{R}^n is $ds_{\mathbb{R}^n}^2 = dr^2 + r^2 ds_{S^{n-1}}^2$, which is to say that metrically speaking $\mathbb{R}^n = Cone(S^{n-1})$. The corresponding radial-horizontal spherical-vertical splitting of the metric on C is

$$(6) \quad d^2 s_C = (dr^2 + r^2 \Sigma(\theta_{in})^2) + r^2 \Sigma_{i < j < n} (\theta_{ij})^2$$

$$(7) \quad = \pi^* ds_{\mathbb{R}^n}^2 + d^2 s_{fiber}$$

which shows the projection $C \rightarrow \mathbb{R}^n$ becomes a Riemannian submersion away from the cone point.

6. MAIN RESULT: MICZ-KEPLER FROM A MECHANICAL SYSTEM ON THE CONE.

A natural mechanical system consists of a configuration space Q , endowed with a Riemannian metric ds_Q^2 and a potential function $V : Q \rightarrow \mathbb{R}$. This data defines a Hamiltonian on T^*Q whose Hamiltonian H is kinetic plus potential: $H = K + V$, where the kinetic energy K is induced by the metric. In standard canonical coordinates $(q, p) = (q_i, p^i)$ for T^*Q we have $K(q, p) = \frac{1}{2} g^{ij}(q) p_i p_j$ if $ds_Q^2 = g_{ij} dq^i dq^j$.

Theorem 1. *Take the canonical metric cone $C = \text{Cone}(SO(n))$ described in the previous section so as to get a Riemannian metric on $C_0 = \text{Cone}(SO(n) \setminus \{0\}) \rightarrow \mathbb{R}^n \setminus \{0\}$. Take potential function $V = -\frac{1}{r} : C_0 \rightarrow \mathbb{R}$ where $r : \text{Cone}(SO(n)) \rightarrow [0, \infty)$ denote the cone's radial coordinate. Then the resulting natural mechanical system (Hamiltonian (8) below) on T^*C_0 is invariant under the lifted action of $SO(n)$, and so is also invariant by $SO(n-1)$. The symplectic reduction of this system by $SO(n-1)$ at any particular $\mu \in \mathfrak{so}(n-1)^* \cong \mathfrak{so}(n-1)$ yields the generalized MICZ-Kepler system associated to the adjoint orbit through μ .*

We see from the expression (2) for the metric on the cone that the Hamiltonian of this theorem is

$$(8) \quad H = \frac{1}{2}(p_r^2 + \frac{1}{r^2}\Sigma \xi_{ij}^2) - \frac{1}{r}$$

where r, p_r are canonical coordinates on $T^*\mathbb{R}^+$ and the ξ_{ij} are the Lie-Poisson coordinates – linear coordinates on $\mathfrak{so}(n)^*$ – induced by the choice of basis $e_i \wedge e_j$ for $\mathfrak{so}(n)$.

Corollary 1. *Any solution $(q(t), \phi(t))$ to the generalized MICZ-Kepler system can be constructed as follows. Fix $\xi \in \mathfrak{so}(n), w \in S^{n-1}$. Fix a solution $r(t)$ to the 1-dimensional Kepler problem: $\ddot{r} = -V'_\mu(r)$ where the effective potential is $V_\mu(r) = -\frac{1}{r} + \frac{\mu^2}{r^2}$ with $\mu^2 = |\xi|^2$, and a solution $u(t)$ to the ODE $\dot{u} = \frac{1}{r(t)^2}$. Then*

$$(9) \quad q(t) = r(t)\exp(u(t)\xi)w$$

The adjoint bundle variable $\phi(t)$ is obtained by parallel translating an initial adjoint vector $\phi(0) = [f, A_f(\xi)]$ along $q(t)$, where $f \in C_0$ is any element projecting to $q(0)$ and A is the connection one-form (3).

Special Cases.

1. Kepler. Take ξ horizontal over the initial $q(0)$, so that $A(\xi) = 0$ and $\phi(t) = 0$. For simplicity, take the initial $q(0)$ in the direction $w = e_n$. Then horizontality implies $\xi = \Sigma v_i e_i \wedge e_n = \vec{v} \wedge e_n$ is an infinitesimal rotation in the \vec{v}, e_n plane. The solution $q(t)$ then lies in this plane. Set $\theta(t) = u(t)|\xi|$. Then $(r(t), \theta(t))$ is a solution to Kepler's equations expressed in polar coordinates.

2. Magnetic cone. Meng takes his $\phi(0)$ to be of “magnetic type”, which means, relative to a local trivialization, that $\phi(0)^2 = -\mu Id$. In other words, up to scale $\phi(t)$ is an almost complex structure on \mathbb{R}^n , compatible to the standard complex structure. Thus ξ satisfies $\xi^2 = -\mu^2 Id$. Set $J = \xi/\mu$ so that J is an honest almost complex structure. Then $\exp(u\xi) = \cos(\mu u)I + \sin(\mu u)J$. The solution $q(t)$ lies on the two-plane spanned by $q(0)$ and $Jq(0)$. Indeed, it is another Keplerian conic on that plane, as Meng showed in [3].

3. Generic. Take ξ generic, meaning that it has $[n/2]$ distinct nonzero eigenvalues $\pm i\omega_j$ linearly independent over the rationals. We can, by conjugating by a rotation, put ξ into the normal form $\Sigma \omega_j e_{2j-1} \wedge e_{2j}$. Then $\theta \mapsto \exp(\theta\xi)$ is a dense curve on a standard maximal torus in $SO(n)$. For negative energy the corresponding one dimensional Kepler motion is periodic with period T , and without collision (since $\xi \neq 0$). We can arrange that $2\pi/T$ is rationally independent of the ω_j . Then the corresponding solution curve $q(t)$ forms a dense winding on a kind of “annular projection” $r_{min} \leq r \leq r_{max}$ to \mathbb{R}^n of a torus of dimension $[n/2] + 1$.

7. PROOF OF THEOREM 1.

We apply the general theory of reduction of cotangent bundles of a principal bundle. We first describe that general theory. See [5], particularly pp. 160-163, or the earlier references [5, 8, 7] for perhaps more leisurely descriptions.

Let $G \rightarrow Q \rightarrow S$ be a principal G -bundle. G acts on T^*Q with G -equivariant momentum map $J : T^*Q \rightarrow \mathfrak{g}^*$. The quotient $(T^*Q)/G$ is naturally a Poisson manifold whose symplectic leaves are the symplectic reduced spaces $J^{-1}(\mathcal{O}_\mu)/G = J^{-1}(\mu)/G_\mu$ where $\mathcal{O}_\mu = G \cdot \mu$ denotes the coadjoint orbit through $\mu \in \mathfrak{g}^*$. The general theory proceeds by using a connection on Q to define a symplectic isomorphism with these reduced spaces.

Differentiate the sequence of maps $G \longrightarrow Q \xrightarrow{\pi} S$ at fixed $q \in Q$ to obtain

the sequence of linear maps $\mathfrak{g} \xrightarrow{\sigma_q} TQ \xrightarrow{d\pi_q} T_s S$ where $s = \pi(q)$. The sequence is exact: the image of σ_q equals the kernel of $d\pi_q$. (This common image is called the vertical space at q .) Letting q vary parametrically we obtain the ‘Atiyah sequence’ (described in [1])

$$(10) \quad Q \times \mathfrak{g} \xrightarrow{\sigma} TQ \xrightarrow{d\pi} \pi_S^* TS$$

which is an exact sequence of G -equivariant vector bundles over Q . Dualizing the first map of (10), and composing with the projection yields the momentum map :

$$J : T^*Q \xrightarrow{\sigma^*} Q \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^* .$$

A connection A for $Q \rightarrow S$ induces a G -invariant splitting of (10) and hence a G -equivariant isomorphism:

$$TQ \cong \pi_S^* TS \oplus (Q \times \mathfrak{g}).$$

Dualizing yields the G -equivariant isomorphism:

$$(11) \quad T^*Q \cong \pi_S^* T^*S \oplus (Q \times \mathfrak{g}^*).$$

Now G acts on the bundle $Q \times \mathfrak{g}$ by $g(q, \xi) = (qg, Ad_{g^{-1}}\xi)$ as per the equivalence relation used to define the adjoint bundle and thus the action of G on $Q \times \mathfrak{g}^*$ is the one used to define the co-adjoint bundle. Forming the quotient by G we thus get the bundle isomorphism

$$\Psi_A : (T^*Q)/G \cong T^*S \oplus Ad^*(Q)$$

over S which is our desired identification. We refer to the right hand side of this isomorphism as being “on the Adjoint bundle side” in what follows.

From our factorization of J we see that under the isomorphism Ψ_A the reduced spaces $J^{-1}(\mu)/G_\mu = J^{-1}(\mathcal{O}_\mu)/G$ become the submanifolds $T^*S \oplus (\mathcal{O}_\mu)(Q)$, which are the Adjoint bundle phase spaces and the Sternberg phase spaces of ([3]). If \mathfrak{g} is endowed with a bi-invariant Killing form as above, then the co-adjoint orbit bundle $(\mathcal{O}_\mu)(Q)$ is identified with a corresponding adjoint orbit.

We need more detail regarding the isomorphism Ψ_A and the Poisson brackets on the universal phase spaces in order to pull-back the cone Hamiltonian (8) and compute equations of motion. The connection defines horizontal lift operators $h_q : T_s S \rightarrow T_q Q$, $q \in Q$, which are linear operators whose image is the horizontal space $\mathcal{H}_q = \ker(A(q)) \subset T_q Q$ of the splitting of TQ . The dual of h_q is $h_q^* : T_q^* Q \rightarrow T_s^* S$

and is one factor of the isomorphism (11). Write $[q, P]$ for the equivalence class in T^*Q/G of $(q, P) \in T^*Q$. Then

$$\Psi_A([q, P]) = (\pi(q), h_q^*P) \oplus [q, J(q, P)] \in T^*S \oplus \text{Ad}^*(Q).$$

We describe Ψ_A in coordinates. Let $(x^i, g) \in \mathbb{R}^d \times G$ be coordinates on Q induced by a local trivialization $Q_U \cong U \times G$ of $Q \rightarrow S$, together with coordinates on $U \subset S$. Then, over U we have $T^*Q \cong T^*U \times T^*G = T^*U \times G \times \mathfrak{g}^*$ with coordinates (x^i, p_i, g, ξ_a) where the coordinates ξ_a are Lie-Poisson linear coordinates on \mathfrak{g}^* relative to a basis e_a of \mathfrak{g} . Thus $(T^*Q)/G \cong T^*U \times \mathfrak{g}^*$ with coordinates x_i, p_i, ξ_a . On the other hand, the same data (x^i, g) and basis e_a yield coordinates x_i, π_i, ξ_a for $T^*S \oplus \text{Ad}^*(Q)$. Relative to these two sets of coordinates the map Ψ_A is the minimal coupling procedure $\Psi_A(x, p_i, \xi) = (x^i, p_i - \xi_a A_i^a(x), \xi_a) = (x^i, \pi_i, \xi_a)$ where $A(x) = \Sigma e_a A_i^a(x) dx$ is the connection one-form relative to the local trivialization and coordinates. The brackets on the Adjoint bundle side are $\{x_i, \pi_j\} = \delta_j^i$, $\{\xi_a, \xi_b\} = -c_{ab}^d \xi_d$ and $\{\pi_i, \pi_j\} = -\xi_a F_{ij}^a$, $\{\pi_i, \xi_a\} = D_i \xi^a = [A_i, \xi]_a$. Here F_{ij}^a is the expression for the curvature of A in this local trivialization. (Compare eqs (12.2) of [5] to (3.2) [3]. Note that in the triple of displayed equations immediately following (12.2) of [5] most terms should have a capital P immediately in front of them.) These agree with the brackets found in Meng for the case of $so(n-1)$.

Now we recompute the Hamiltonian (8) on the Adjoint bundle side using Ψ_A . The dual of the metric splitting (6) yields

$$(12) \quad K_C = \frac{1}{2} p_r^2 + \frac{1}{r^2} (\Sigma_{i=1}^n \xi_{in}^2 + \frac{1}{2} \Sigma_{i < j < n} \xi_{ij}^2)$$

$$(13) \quad = \frac{1}{2} p_r^2 + \frac{1}{r^2} (h^* K_{S^n} + |\phi|^2)$$

where $h^* : T^*SO(n) \rightarrow T^*S^{n-1}$ is the connection induced dual of the horizontal lift. The fiber term $\Sigma_{i < j < n} \xi_{ij}^2$ corresponds, on the Adjoint bundle side, to the Casimir function $|\phi|^2$ of $SO(n-1)$, viewed as a function on the adjoint bundle. So the Hamiltonian, viewed on adjoint bundle side, reads

$$H = \frac{1}{2} (p_r^2 + \frac{1}{r^2} K_{S^n} + \frac{1}{r^2} |\phi|^2) - \frac{1}{r}$$

The sum of the first two terms $\frac{1}{2} (p_r^2 + \frac{1}{r^2} K_{S^n})$ is the usual kinetic energy $\frac{1}{2} \Sigma_{i=1}^n \pi_i^2$ on \mathbb{R}^n written in spherical variables. Thus

$$(14) \quad H = \frac{1}{2} (\Sigma \pi_i^2 + \frac{1}{r^2} |\phi|^2) - \frac{1}{r}$$

which is the MICZ-Kepler Hamiltonian.

8. PROOF OF THE COROLLARY.

We compute the equations of motion on C_0 , using the expression (8) for the Hamiltonian and the fact that $\Omega = \|\xi\|^2$ is a Casimir. Set $\mu^2 = \Omega$ and $V_\mu(r) = -\frac{1}{r} + \frac{\mu^2}{r^2}$. Then Hamilton's equations on $T^*C_0 = T^*\mathbb{R}^+ \times T^*SO(n) = \mathbb{R}^+ \times \mathbb{R} \times SO(n) \times so(n)$ are:

$$\begin{aligned} \dot{r} &= \{r, H\} = p_r \\ \dot{p}_r &= \{p_r, H\} = -V'_\mu(r) \\ \dot{g} &= g \frac{\partial H}{\partial \xi} = g \frac{1}{r^2} \xi \end{aligned}$$

$$\dot{\xi}_a = \{\xi_a, H\} = \frac{1}{2r^2} \{\xi_a, \Omega\} = 0$$

The first pair of equations decouple from the second pair, and assert that (r, p_r) evolves as per the one-dimensional radial Kepler equation with effective potential V_μ . The last equation asserts that $\xi \in so(n)$ is constant. The equation for g asserts that $g(t) = g_0 \exp(u(t)\xi)$ where $du/dt = 1/r^2$. Indeed, the solution to $\dot{g} = g\xi$ through $g = Id$ is the one-parameter subgroup $\exp(t\xi)$ and this flow is generated by the Hamiltonian $\frac{1}{2}\Omega$. We have scaled the Hamiltonian on $SO(n)$ by the (time-dependent) factor $1/r^2$ and used left-invariance. To rewrite $g_0 \exp(u(t)\xi) = \exp(u(t)\tilde{\xi})g_0$ we can set $\tilde{\xi} = g_0 \xi g_0^{-1}$.

We now have the solution the Kepler equation on the cone: $(r(t), g(t))$ with $g(t) = \exp(u(t)\xi)g_0$. Recall the bundle projection is $(r, g) \mapsto rge_n$ and use that any unit vector w can be written $g_0 e_n$ to the expression for $q(t)$ in the corollary. QED

9. OTHER GROUPS

The tricks used here apply to any Lie group G in place of $SO(n)$ provided that G is endowed with an faithful orthogonal representation on \mathbb{R}^n which is transitive on the unit sphere S^{n-1} . We get the theorem that the Kepler problem on $Cone(G)$ is equivalent to the ‘MICZ-Kepler- G ’ problem whose ‘color variables’ lie in an adjoint orbit bundle for G over $\mathbb{R}^n \setminus \{0\}$. The standard families of such groups are the unitary groups $U(n)$ on $\mathbb{R}^{2n} = \mathbb{C}^n$ and $Sp(n; \mathbb{H})$ on $\mathbb{H}^n = \mathbb{R}^{4n}$.

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